3

Analyzing Solutions

3.1 Economic Analysis of Solution Reports

A substantial amount of interesting economic information can be gleaned from the solution report of a model. In addition, optional reports, such as range analysis, can provide further information. The usual use of this information is to do a quick "what if" analysis. The typical kinds of what if questions are:

    (a) What would be the effect of changing a capacity or demand?

    (b) What if a new opportunity becomes available? Is it a worthwhile opportunity?

3.2 Economic Relationship Between Dual Prices and Reduced Costs

The reader hungering for unity in systems may convince himself or herself that a reduced cost is really a dual price born under the wrong sign. Under our convention, the reduced cost of a variable x is really the dual price with the sign reversed on the constraint $x \geq 0$. Recall the reduced cost of the variable x measures the rate at which the solution value deteriorates as x is increased from zero. The dual price on $x \geq 0$ measures the rate at which the solution value improves as the right-hand side (and thus x) is increased from zero.

Our knowledge about reduced costs and dual prices can be restated as:

> *Reduced cost of an (unused) activity*: amount by which profits will decrease if one unit of this activity is forced into the solution.

> *Dual price of a constraint*: one unit reduces amount by which profits will decrease if the availability of the resource associated with this constraint.

We shall argue and illustrate that the reduced cost of an activity is really its net opportunity cost if we "cost out" the activity using the dual prices as charges for resource usage. This sounds like good economic sense. If one unit of an activity is forced into the solution, it effectively reduces the availability of the resources it uses. These resources have an imputed value by way of the dual prices. Therefore, the activity should be charged for the value used. Let's look at an example and check if the argument works.

3.2.1 The Costing Out Operation: An Illustration

Suppose Enginola is considering adding a video recorder to its product line. Market Research and Engineering estimate the direct profit contribution of a video recorder as $47 per unit. It would be manufactured on the Astro line and would require 3 hours of labor. If it is produced, it will force the reduction of both Astro production (because it competes for a production line) and Cosmo production (because it competes for labor). Is this tradeoff worthwhile? It looks promising. The video recorder makes more dollars per hour of labor than a Cosmo and it makes more efficient use of Astro capacity than Astros. Recall the dual prices on the Astro and labor capacities in the original solution were $5 and $15. If we add this variable to the model, it would have a +47 in the objective function, a +1 in row 2 (the Astro capacity constraint), and a +3 in row 4 (the labor capacity constraint). We can "cost out" an activity or decision variable by charging it for the use of scarce resources. What prices should be charged? The obvious prices to use are the dual prices. The +47 profit contribution can be thought of as a negative cost. The costing out calculations can be arrayed as in the little table below:

Row	Coefficient	Dual Price	Charge
1	−47	1	−47
2	1	+5	+5
3	0	0	0
4	3	15	+45
		Total opportunity cost =	+3

Thus, a video recorder has an opportunity cost of $3. A negative one (−1) is applied to the 47 profit contribution because a profit contribution is effectively a negative cost. The video recorder's net cost is positive, so it is apparently not worth producing.

The analysis could be stopped at this point, but out of curiosity we'll formulate the relevant LP and solve it. If V = number of video recorders to produce, then we wish to solve:

```
MAX = 20 * A + 30 * C + 47 * V;
      A                  + V <=  60;
                  C            <=  50;
      A  + 2 * C  + 3 * V <= 120;
```

The solution is:

```
Optimal solution found at step:          1
Objective value:                   2100.000

Variable            Value          Reduced Cost
     A          60.000000            0.000000
     C          30.000000            0.000000
     V           0.000000            3.000000

    Row     Slack or Surplus      Dual Price
     1        2100.000000           1.000000
     2           0.000000           5.000000
     3          20.000000           0.000000
     4           0.000000          15.000000
```

Video recorders are not produced. Notice the reduced cost of V is $3, the value we computed when we "costed out" V. This is an illustration of the following relationship:

> The reduced cost of an activity equals the weighted sum of its resource usage rates minus its profit contribution rate, where the weights applied are the dual prices. A

"min" objective is treated as having a dual price of +1. A "max" objective is treated
as having a dual price of −1 in the costing out process.

Notice that the dual prices of an LP fully allocate the total profit to all the scarce resources, i.e., for the
above example, 5 *60 + 0*50 + 15*120 = 2100.

3.2.2 Dual Prices, LaGrange Multipliers, KKT Conditions, and Activity Costing

When you solve a continuous optimization problem with LINGO or What's*Best!*, you can optionally
have dual prices reported for each constraint. For simplicity, assume that our objective is to maximize
and all constraints are less-than-or-equal-to when all variable expressions are brought to the left-hand
side. The dual price of a constraint is then the rate of change of the optimal objective value with
respect to the right-hand side of the constraint. This is a generalization to inequality constraints of the
idea of a LaGrange multiplier for equality constraints. This idea has been around for more than 100
years. To illustrate, consider the following slightly different, nonlinear problem:

```
[ROW1] MAX = 40*( X+1)^.5 + 30*( Y+1)^.5 + 25*( Z+1)^.5;
[ROW2]          X                        + 15* Z    <= 45;
[ROW3]                          Y              + Z    <= 45;
[ROW4]          X* X      + 3* Y*Y      + 9 * Z*Z <= 3500;
```

We implicitly assume that $X, Y, Z >= 0$.

When solved, you get the solution:

```
Objective value =   440.7100
Variable            Value        Reduced Cost
      X          45.00000          0.0000000
      Y          22.17356          0.0000000
      Z          0.0000000         0.1140319

    Row     Slack or Surplus     Dual Price
    ROW2       0.0000000         0.8409353
    ROW3       22.82644          0.0000000
    ROW4       0.0000000         0.02342115
```

For example, the dual price of .8409353 on *ROW2* implies that if the RHS of *ROW2* is increased by a
small amount, epsilon, the optimal objective value will increase by about .8409353 * epsilon.

When trying to understand why a particular variable or activity is unused (i.e., at zero), a useful
perspective is that of "costing out the activity". We give the variable "credit" for its incremental
contribution to the objective and charge it for its incremental usage of each constraint, where the
charging rate applied is the dual price of the constraint. The incremental contribution, or usage, is
simply the partial derivative of the LHS with respect to the variable. The costing out of variable *Z* is
illustrated below:

```
Row       Partial w.r.t Z      Dual price      Total charge
ROW1           12.5                -1             -12.5
ROW2           15              .8409353         12.614029
ROW3           1                  0                0
ROW4           0              .02342115           0
                             Net(Reduced Cost): .11403
```

On the other hand, if we do the same costing out for *X*, we get:

Row	Partial w.r.t X	Dual price	Total charge
ROW1	2.9488391	-1	-2.9488391
ROW2	1	.8409353	.8409353
ROW3	0	0	0
ROW4	90	.02342115	2.107899
		Net(Reduced Cost):	0

These two computations are illustrations of the Karush/Kuhn/Tucker (KKT) conditions, namely, in an optimal solution:

> a) a variable that has a positive reduced cost will have a value of zero;
> b) a variable that is used (i.e., is strictly positive) will have a reduced cost of zero;
> c) a "<=" constraint that has a positive dual price will have a slack of zero;
> d) a "<=" constraint that has strictly positive slack, will have a dual price of zero.

These conditions are sometimes also called complementary slackness conditions.

3.3 Range of Validity of Reduced Costs and Dual Prices

In describing reduced costs and dual prices, we have been careful to limit the changes to "small" changes. For example, if the dual price of a constraint is $3/hour, then increasing the number of hours available will improve profits by $3 for each of the first few hours (possibly less than one) added. However, this improvement rate will generally not hold forever. We might expect that, as we make more hours of capacity available, the value (i.e., the dual price) of these hours would not increase and might decrease. This might not be true for all situations, but for LP's it is true that increasing the right-hand side of a constraint cannot cause the constraint's dual price to increase. The dual price can only stay the same or decrease.

As we change the right-hand side of an LP, the optimal values of the decision variables may change. However, the dual prices and reduced costs will not change as long as the "character" of the optimal solution does not change. We will say the character changes (mathematicians say the basis changes) when either the set of nonzero variables or the set of binding constraints (i.e., have zero slack) changes. In summary, as we alter the right-hand side, the same dual prices apply as long as the "character" or "basis" does not change.

Most LP programs will optionally supplement the solution report with a range (i.e., sensitivity analysis) report. This report indicates the amounts by which individual right-hand side or objective function coefficients can be changed unilaterally without affecting the character or "basis" of the optimal solution. Recall the previous model:

```
MAX = 20 * A + 30 * C + 47 * V;
        A              + V <=  60;
                C          <=  50;
        A  + 2 * C  + 3 * V <= 120;
```

To obtain the sensitivity report, while in the window with the program, choose *Range* from the *LINGO* menu. The sensitivity report for this problem appears below:

```
Ranges in which the basis is unchanged:
               Objective Coefficient Ranges
             Current      Allowable      Allowable
Variable   Coefficient     Increase       Decrease
    A       20.00000       INFINITY       3.000000
    C       30.00000       10.00000       3.000000
    V       47.00000       3.000000       INFINITY
                  Right-hand Side Ranges
   Row       Current      Allowable      Allowable
               RHS         Increase       Decrease
    2       60.00000       60.00000       40.00000
    3       50.00000       INFINITY       20.00000
    4       120.0000       40.00000       60.00000
```

Again, we find two sections, one for variables and the second for rows or constraints. The 3 in the *A* row of the report means the profit contribution of *A* could be decreased by up to $3/unit without affecting the optimal amount of *A* and *C* to produce. This is plausible because one Astro and one Cosmo together make $50 of profit contribution. If the profit contribution of this pair is decreased by $3 (to $47), then a *V* would be just as profitable. Note that one *V* uses the same amount of scarce resources as one Astro and one Cosmo together. The *INFINITY* in the same section of the report means increasing the profitability of *A* by any positive amount would have no effect on the optimal amount of *A* and *C* to produce. This is intuitive because we are already producing *A*'s to their upper limit.

The "allowable decrease" of 3 for variable *C* follows from the same argument as above. The allowable increase of 10 in the *C* row means the profitability of *C* would have to be increased by at least $10/unit (thus to $40/unit) before we would consider changing the values of *A* and *C*. Notice at $40/unit for *C*'s, the profit per hour of labor is the same for both *A* and *C*.

In general, if the *objective function coefficient* of a single variable is *changed within the range* specified in the first section of the range report, then the optimal values of the *decision variables*, *A*, *C*, and *V*, in this case, *will not change*. The dual prices, reduced cost and profitability of the solution, however, may change.

In a complementary sense, if the *right-hand side* of a single constraint is *changed within the range* specified in the second section of the range report, then the *optimal values* of the dual prices and reduced costs *will not change*. However, the values of the decision variables and the profitability of the solution may change.

For example, the second section tells us that, if the right-hand side of row 3 (the constraint $C \le 50$) is decreased by more than 20, then the dual prices and reduced costs will change. The constraint will then be $C \le 30$ and the character of the solution changes in that the labor constraint will no longer be binding. The right-hand side of this constraint ($C \le 50$) could be increased an infinite amount, according to the range report, without affecting the optimal dual prices and reduced costs. This makes sense because there already is excess capacity on the Cosmo line, so adding more capacity should have no effect.

Let us illustrate some of these concepts by re-solving our three-variable problem with the amount of labor reduced by 61 hours down to 59 hours. The formulation is:

```
MAX = 20 * A + 30 * C + 47 * V;
        A              + V <= 60;
                C            <= 50;
        A + 2 * C + 3 * V <= 59;
```

The solution is:

```
Optimal solution found at step:        1
Objective value:                 1180.000
Variable            Value          Reduced Cost
     A            59.00000            0.0000000
     C             0.0000000         10.00000
     V             0.0000000         13.00000

   Row     Slack or Surplus      Dual Price
    1          1180.000           1.000000
    2          1.000000           0.0000000
    3          50.00000           0.0000000
    4          0.0000000          20.00000

Ranges in which the basis is unchanged:
                 Objective Coefficient Ranges
              Current       Allowable      Allowable
Variable    Coefficient     Increase       Decrease
    A         20.00000       INFINITY       4.333333
    C         30.00000       10.00000       INFINITY
    V         47.00000       13.00000       INFINITY

                 Right-hand Side Ranges
   Row        Current       Allowable      Allowable
                RHS         Increase       Decrease
    2         60.00000       INFINITY       1.000000
    3         50.00000       INFINITY       50.00000
    4         59.00000       1.000000       59.00000
```

First, note that, with the reduced labor supply, we no longer produce any Cosmos. Their reduced cost is now $10/unit, which means, if their profitability were increased by $10 to $40/unit, then we would start considering their production again. At $40/unit for Cosmos, both products make equally efficient use of labor.

Also note, since the right-hand side of the labor constraint has reduced by more than 60, most of the dual prices and reduced costs have changed. In particular, the dual price or marginal value of labor is now $20 per hour. This is because an additional hour of labor would be used to produce one more $20 Astro. You should be able to convince yourself the marginal value of labor behaves as follows:

Labor Available	Dual Price	Reason
0 to 60 hours	$20/hour	Each additional hour will be used to produce one $20 Astro.
60 to 160 hours	$15/hour	Each additional hour will be used to produce half a $30 Cosmo.
160 to 280 hours	$13.5/hour	Give up half an Astro and add half of a V for profit of 0.5 (–20 + 47).
More than 280 hours	$0	No use for additional labor.

In general, the dual price on any constraint will behave in the above stepwise decreasing fashion.

Figures 3.1 and 3.2 give a global view of how total profit is affected by changing either a single objective coefficient or a single right-hand side. The artists in the audience may wish to note that, for a maximization problem:

a) Optimal total profit as a function of a single objective coefficient always has a bowl shape. Mathematicians say it is a convex function.

b) Optimal total profit as a function of a single right-hand side value always has an inverted bowl shape. Mathematicians say it is a concave function.

For some problems, as in Figures 3.1 and 3.2, we only see half of the bowl. For minimization problems, the orientation of the bowl in (a) and (b) is simply reversed.

When we solve a problem for a particular objective coefficient or right-hand side value, we obtain a single point on one of these curves. A range report gives us the endpoints of the line segment on which this one point lies.

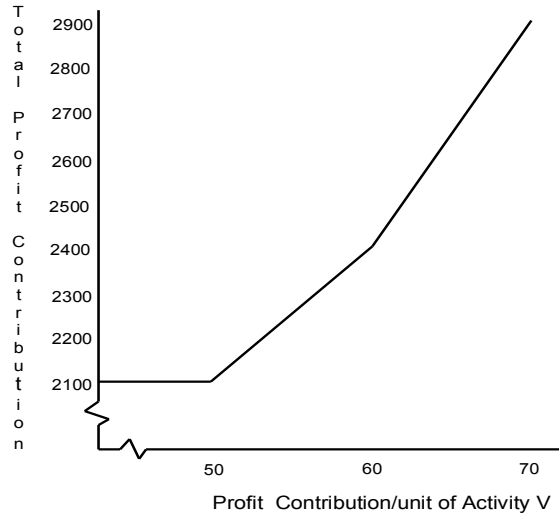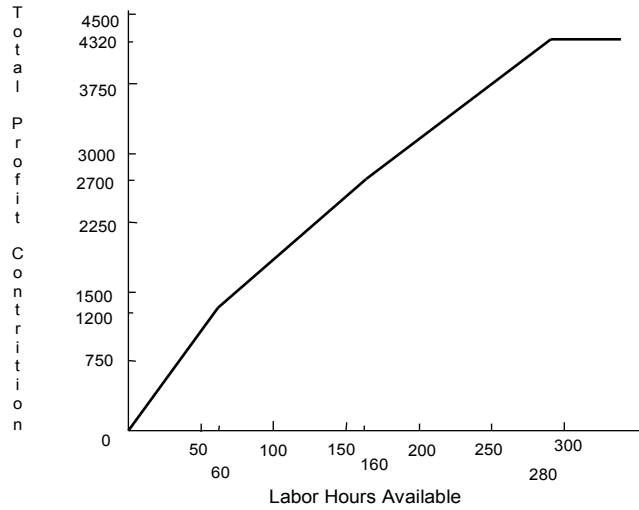Figure 3.1 Total Profit vs. Profit Contribution per Unit of Activity *V*

Profit Contribution/unit of Activity V

Figure 3.2 Profit vs. Labor Available

Labor Hours Available

3.3.1 Predicting the Effect of Simultaneous Changes in Parameters—The 100% Rule

The information in the range analysis report tells us the effect of changing a single cost or resource parameter. The range report for the Enginola problem is presented as an example:

```
Ranges in which the basis is unchanged:
                    Objective Coefficient Ranges
               Current        Allowable        Allowable
Variable     Coefficient       Increase         Decrease
      A        20.00000        INFINITY         5.000000
      C        30.00000        10.00000         30.00000

                    Right-hand Side Ranges
     Row         Current        Allowable        Allowable
                     RHS         Increase         Decrease
       2        60.00000        60.00000         40.00000
       3        50.00000        INFINITY         20.00000
       4        120.0000        40.00000         60.00000
```

The report indicates the profit contribution of an Astro could be decreased by as much as $5/unit without changing the basis. In this case, this means that the optimal solution would still recommend producing 60 Astros and 30 Cosmos.

Suppose, in order to meet competition, we are considering lowering the price of an Astro by $3/unit and the price of a Cosmo by $10/unit. Will it still be profitable to produce the same mix? Individually, each of these changes would not change the solution because $3 < 5$ and $10 < 30$.

However, it is not clear these two changes can be made simultaneously. What does your intuition suggest as a rule describing the simultaneous changes that do not change the basis (mix)?

The 100% Rule. You can think of the allowable ranges as slack, which may be used up in changing parameters. It is a fact that any combination of changes will not change the basis if the sum of percentages of slack used is less than 100%. For the simultaneous changes we are contemplating, we have:

$$\left(\frac{3}{5}\right) \times 100 + \left(\frac{10}{30}\right) \times 100 = 60\% + 33\% = 93.3\% < 100\%$$

This satisfies the condition, so the changes can be made without changing the basis. Bradley, Hax, and Magnanti (1977) have dubbed this rule the 100% rule. Since the value of *A* and *C* do not change, we can calculate the effect on profits of these changes as $-3 \times 60 - 10 \times 30 = -480$. So, the new profit will be $2100 - 480 = 1620$.

The altered formulation and its solution are:

```
MAX  = 17 * A + 20 * C;
            A             <=  60;
                     C <= 50;
            A +  2 * C <= 120;
```

```
Optimal solution found at step:        1
Objective value:                  1620.000
Variable            Value         Reduced Cost
    A            60.00000          0.0000000
    C            30.00000          0.0000000
    Row     Slack or Surplus       Dual Price
    1           1620.000            1.000000
    2          0.0000000            7.000000
    3           20.00000            0.0000000
    4          0.0000000            10.00000
```

3.4 Sensitivity Analysis of the Constraint Coefficients

Sensitivity analysis of the right-hand side and objective function coefficients is somewhat easy to understand because the objective function value changes linearly with modest changes in these coefficients. Unfortunately, the objective function value may change nonlinearly with changes in constraint coefficients. However, there is a very simple formula for approximating the effect of small changes in constraint coefficients. Suppose we wish to examine the effect of decreasing by a small amount e the coefficient of variable j in row i of the LP. The formula is:

(improvement in objective value) ≈ (value of variable j) × (dual price of row i) × e

Example: Consider the problem:

```
MAX = (20 * A) + (30 * C);
A <= 65;
C <= 50;
A + 2 * C <= 115;
```

with solution:

```
Optimal solution found at step:        1
Objective value:                  2050.000
Variable            Value         Reduced Cost
    A            65.00000          0.0000000
    C            25.00000          0.0000000
    Row     Slack or Surplus       Dual Price
    1           2050.000            1.000000
    2          0.0000000            5.000000
    3           25.00000            0.0000000
    4          0.0000000            15.00000
```

Now, suppose it is discovered that the coefficient of C in row 4 should have been 2.01, rather than 2. The formula implies the objective value should be decreased by approximately $25 \times 15 \times .01 = 3.75$.

The actual objective value, when this altered problem is solved, is 2046.269, so the actual decrease in objective value is 3.731.

The formula for the effect of a small change in a constraint coefficient makes sense. If the change in the coefficient is small, then the values of all the variables and dual prices should remain essentially unchanged. So, the net effect of changing the 2 to a 2.01 in our problem is effectively to try to use $25 \times .01$ additional hours of labor. So, there is effectively $25 \times .01$ fewer hours available. However, we have seen that labor is worth \$15 per hour, so the change in profits should be about $25 \times .01 \times 15$, which is in agreement with the original formula.

This type of sensitivity analysis gives some guidance in identifying which coefficient should be accurately estimated. If the product of variable *j*'s value and row *i*'s dual price is relatively large, then the coefficient in row *i* for variable *j* should be accurately estimated if an accurate estimate of total profit is desired.

3.5 The Dual LP Problem, or the Landlord and the Renter

As you formulate models for various problems, you will probably discover that there are several rather different-looking formulations for the same problem. Each formulation may be correct and may be based on taking a different perspective on the problem. An interesting mathematical fact is, for LP problems, there are always two formulations (more accurately, a multiple of two) to a problem. One formulation is arbitrarily called the primal and the other is referred to as the dual. The two different formulations arise from two different perspectives one can take towards a problem. One can think of these two perspectives as the landlord's and the renter's perspectives.

In order to motivate things, consider the following situations. Some textile "manufacturers" in Italy own no manufacturing facilities, but simply rent time as needed from firms that own the appropriate equipment. In the U.S., a similar situation exists in the recycling of some products. Firms that recycle old telephone cable may simply rent time on the stripping machines that are needed to separate the copper from the insulation. This rental process is sometimes called "tolling". In the perfume industry, many of the owners of well-known brands of perfume own no manufacturing facilities, but simply rent time from certain chemical formulation companies to have the perfumes produced as needed. The basic feature of this form of industrial organization is that the owner of the manufacturing resources never owns either the raw materials or the finished product.

Now, suppose you want to produce a product that can use the manufacturing resources of the famous Enginola Company, manufacturer of Astros, Cosmos, and Video Recorders. You would thus like to rent production capacity from Enginola. You need to deduce initial reasonable hourly rates to offer to Enginola for each of its three resources: Astro line capacity, Cosmo line capacity, and labor. These three hourly rates are your decision variables. You in fact would like to rent all the capacity on each of the three resources. Thus, you want to minimize the total charge from renting the entire capacities (60, 50, and 120). If your offer is to succeed, you know your hourly rates must be sufficiently high, so none of Enginola's products are worth producing (e.g., the rental fees foregone by producing an Astro should be greater than 20). These "it's better to rent" conditions constitute the constraints.

Formulating a model for this problem, we define the variables as follows:

PA = price per unit to be offered for Astro line capacity,
PC = price per unit to be offered for Cosmo line capacity,
PL = price per unit to be offered for labor capacity.

Then, the appropriate model is:

The Dual Problem:
```
MIN = 60 * PA + 50 * PC + 120 * PL;
!ASTRO; PA + PL > 20;
!COSMO; PC + 2*PL > 30;
!VR;    PA + 3 * PL > 47;
```

The three constraints force the prices to be high enough, so it is not profitable for Enginola to produce any of its products.

The solution is:

```
Optimal solution found at step:        2
Objective value:                 2100.000
Variable            Value          Reduced Cost
      PA          5.000000            0.0000000
      PC          0.0000000          20.00000
      PL          15.00000            0.0000000

      Row     Slack or Surplus      Dual Price
       1          2100.000           1.000000
       2          0.0000000         -60.00000
       3          0.0000000         -30.00000
       4          3.000000            0.0000000
```

Recall the original, three-product Enginola problem was:

The Primal Problem:
```
MAX = 20 * A + 30 * C + 47 * V;
A +               V <=   60;
        C               <=   50;
A + 2 * C + 3 * V <= 120;
```

with solution:

```
Optimal solution found at step:        1
Objective value:                 2100.000
Variable            Value          Reduced Cost
       A          60.00000            0.0000000
       C          30.00000            0.0000000
       V          0.0000000           3.000000

      Row     Slack or Surplus      Dual Price
       1          2100.000           1.000000
       2          0.0000000           5.000000
       3          20.00000            0.0000000
       4          0.0000000          15.00000
```

Notice the two solutions are essentially the same, except prices and decision variables are reversed. In particular, note the price the renter should pay is exactly the same as Enginola's original profit contribution. This "Minimize the rental cost of the resources, subject to all activities being unprofitable" model is said to be the dual problem of the original "Maximize the total profit, subject to not exceeding any resource availabilities" model. The equivalence between the two solutions shown above always holds. Upon closer scrutiny, you should also notice the dual formulation is essentially the primal formulation "stood on its ear," or its transpose, in fancier terminology.

Why might the dual model be of interest? The computational difficulty of an LP is approximately proportional to m^2n, where m = number of rows and n = number of columns. If the number of rows in the dual is substantially smaller than the number of rows in the primal, then one may prefer to solve the dual.

Additionally, certain constraints, such as simple upper bounds (e.g., $x \le 1$) are computationally less expensive than arbitrary constraints. If the dual contains only a small number of arbitrary constraints, then it may be easier to solve the dual even though it may have a large number of simple constraints.

The term "dual price" arose because the marginal price information to which this term is applied is a decision variable value in the dual problem.

We can summarize the idea of dual problems as follows. If the original or primal problem has a Maximize objective with $\leq$ constraints, then its dual has a Minimize objective with $\geq$ constraints. The dual has one variable for each constraint in the primal and one constraint for each variable in the primal. The objective coefficient of the kth variable of the dual is the right-hand side of the kth constraint in the primal. The right-hand side of constraint k in the dual is equal to the objective coefficient of variable k in the primal. Similarly, the coefficient in row i of variable j in the dual equals the coefficient in row j of variable i in the primal.

In order to convert all constraints in a problem to the same type, so one can apply the above, note the following two transformations:

(1) The constraint $2x + 3y = 5$ is equivalent to the constraints $2x + 3y \geq 5$ and $2x + 3y \leq 5$;
(2) The constraint $2x + 3y \geq 5$ is equivalent to $-2x - 3y \leq -5$.

Example: Write the dual of the following problem:

$$\text{Maximize } 4x - 2y$$
$$\text{subject to } 2x + 6y \leq 12$$
$$3x - 2y = 1$$
$$4x + 2y \geq 5$$

Using transformations (1) and (2) above, we can rewrite this as:

$$\text{Maximize } 4x - 2y$$
$$\text{subject to } 2x + 6y \leq 12$$
$$3x - 2y \leq 1$$
$$-3x + 2y \leq -1$$
$$-4x - 2y \leq -5$$

Introducing the dual variables $r, s, t,$ and u, corresponding to the four constraints, we can write the dual as:

$$\text{Minimize } \quad 12r + s - t - 5u$$
$$\text{subject to } \quad 2r + 3s - 3t - 4u \geq 4$$
$$6r - 2s + 2t - 2u \geq -2$$

3.6 Problems

1. The Enginola Company is considering introducing a new TV set, the Quasi. The expected profit contribution is $25 per unit. This unit is produced on the Astro line. Production of one Quasi requires 1.6 hours of labor. Using only the original solution below, determine whether it is worthwhile to produce any Quasi's, assuming no change in labor and Astro line capacity.

    The original Enginola problem with solution is below.

    ```
    MAX = 20 * A + 30 * C;
    A          <= 60;
         C <= 50;
    A + 2 * C <= 120;
    ```

```
Optimal solution found at step:        1
Objective value:                  2100.000

Variable            Value         Reduced Cost
    A             60.00000          0.0000000
    C             30.00000          0.0000000

Row     Slack or Surplus      Dual Price
 1           2100.000          1.000000
 2          0.0000000          5.000000
 3           20.00000          0.0000000
 4          0.0000000          15.00000
```

2. The Judson Corporation has acquired 100 lots on which it is about to build homes. Two styles of homes are to be built, the "*Cape Cod*" and the "*Ranch Home*". Judson wishes to build these 100 homes over the next nine months. During this time, Judson will have available 13,000 man-hours of bricklayer labor and 12,000 hours of carpenter labor. Each *Cape Cod* requires 200 man-hours of carpentry labor and 50 man-hours of bricklayer labor. Each *Ranch Home* requires 120 hours of bricklayer labor and 100 man-hours of carpentry. The profit contribution of a *Cape Cod* is projected to be $5,100, whereas that of a *Ranch Home* is projected at $5,000. When formulated as an LP and solved, the problem is as follows:

```
MAX = 5100 * C + 5000 * R;
      C +        R < 100;
200 * C + 100 * R < 12000;
 50 * C + 120 * R < 13000;

Optimal solution found at step:         0
Objective value:                  502000.0

Variable            Value         Reduced Cost
    C             20.00000          0.0000000
    R             80.00000          0.0000000

Row     Slack or Surplus      Dual Price
 1           502000.0          1.000000
 2          0.0000000          4900.000
 3          0.0000000          1.000000
 4           2400.000          0.0000000
```

Ranges in which the basis is unchanged:

		Objective Coefficient Ranges	
	Current	Allowable	Allowable
Variable	Coefficient	Increase	Decrease
C	5100.000	4900.000	100.0000
R	5000.000	100.0000	2450.000

		Right-hand Side Ranges	
Row	Current	Allowable	Allowable
	RHS	Increase	Decrease
2	100.0000	12.63158	40.00000
3	12000.00	8000.000	2000.000
4	13000.00	INFINITY	2400.000

(a) A gentleman who owns 15 vacant lots adjacent to Judson's 100 lots needs some money quickly and offers to sell his 15 lots for $60,000. Should Judson buy? What assumptions are you making?

(b) One of Judson's salesmen who is a native of Massachusetts feels certain he could sell the *Cape Cods* for $2,000 more each than Judson is currently projecting. Should Judson change its planned mix of homes? What assumptions are inherent in your recommendation?

3. Jack Mazzola is an industrial engineer with the Enginola Company. He has discovered a way of reducing the amount of labor used in the manufacture of a Cosmo TV set from 2 hours per set to 1.92 hours per set by replacing one of the assembled portions of the set with an integrated circuit chip. It is not clear at the moment what this chip will cost. Based solely on the solution report below (i.e., do not solve another LP), answer the following questions:

(a) Assuming labor supply is fixed, what is the approximate value of one of these chips in the short run?

(b) Give an estimate of the approximate increase in profit contribution per day of this change, exclusive of chip cost.

```
MAX = 20 * A + 30 * C;
          A              <= 60;
                    C <= 50;
          A +  2 * C <= 120;
```

```
Optimal solution found at step:         1
Objective value:                    2100.000
```

Variable	Value	Reduced Cost
A	60.00000	0.0000000
C	30.00000	0.0000000

Row	Slack or Surplus	Dual Price
1	2100.000	1.000000
2	0.0000000	5.000000
3	20.00000	0.0000000
4	0.0000000	15.00000

Right-hand Side Ranges

Row	Current RHS	Allowable Increase	Allowable Decrease
2	60.00000	60.00000	40.00000
3	50.00000	INFINITY	20.00000
4	120.0000	40.00000	60.00000

4. The Bug product has a profit contribution of $4100 per unit and requires 4 hours in the foundry department and 2 hours in the assembly department. The SuperBug has a profit contribution of $5900 per unit and requires 2 hours in the foundry and 3 hours in assembly. The availabilities in foundry and assembly are 160 hours and 180 hours, respectively. Each hour used in each of foundry and assembly costs $90 and $60, respectively. The following is an LP formulation for maximizing profit contribution in this situation:

```
MAX = 4100 * B + 5900 * S - 90 * F - 60 * A;
       4 * B +   2 * S -      F            = 0;
       2 * B +   3 * S              - A    = 0;
                                F          <= 160;
                                A <= 180;
```

Following is an optimal solution report printed on a typewriter that skipped some sections of the report.

```
Objective value:
Variable            Value           Reduced Cost
      B                                73.33325
      S            60.00000
      F            120.0000            0.0000000
      A            180.0000            0.0000000

    Row     Slack or Surplus      Dual Price
     1           332400.0          1.000000
     2          0.0000000
     3                             1906.667
     4                             0.0000000
     5          0.0000000          1846.667
```

Fill in the missing parts, using just the available information (i.e., without re-solving the model on the computer).

5. Suppose the capacities in the Enginola problem were: Astro line capacity = 45; labor capacity = 100.

    (a) Allow the labor capacity to vary from 0 to 200 and plot:

    - Dual price of labor as a function of labor capacity.
    - Total profit as a function of labor capacity.

    (b) Allow the profit contribution/unit of Astros to vary from 0 to 50 and plot:

    - Number of Astros to produce as a function of profit/unit.
    - Total profit as a function of profit/unit.

6. Write the dual problem of the following problem:

    Minimize $12q + 5r + 3s$
    subject to $q + 2r + 4s \geq 6$
    $5q + 6r - 7s \leq 5$
    $8q - 9r + 11s = 10$

7. The energetic folks at Enginola, Inc. have not been idle. The R & D department has given some more attention to the proposed digital recorder product (code name R) and enhanced it so much that everyone agrees it could be sold for a profit contribution of $79 per unit. Unfortunately, its production still requires one unit of capacity on both the A(stro) and C(osmo) lines. Even worse, it now requires four hours of labor. The Marketing folks have spread the good word about the Astro and Cosmo products, so a price increase has been made possible. Industrial Engineering has been able to increase the capacity of the two lines. The new ex-marine heading Human Resources has been able to hire a few more good people, so the labor capacity has increased to 135 hours. The net result is that the relevant model is now:

```
MAX = 23 * A + 38 * C + 79 * R;
      A               + R <=  75;
              C       + R <=  65;
      A + 2 * C + 4 * R <= 135;
END
```

Without resorting to a computer, answer the following questions, supporting each answer with a one- or two-sentence economic argument that might be understood by your spouse or "significant other."

    (a) How many A's should be produced?
    (b) How many C's should be produced?
    (c) How many R's should be produced?
    (d) What is the marginal value of an additional hour of labor?
    (e) What is the marginal value/unit of additional capacity on the A line?
    (f) What is the marginal value per unit of additional capacity on the C line?